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1. Learning Outcomes

After studying this module, you shall be able to

- Understand the difference between an ideal classical gas and an ideal Bose gas arising because of indistinguishability of particles leading to quantum effects not observed classically.
- See that there are two kinds of Bose Gas, with zero mass and zero chemical potential and the other with non-zero mass with chemical potential less than zero.
- Understand the relevance of mean thermal wavelength or thermal de Broglie wavelength as a deciding parameter to characterize when are the quantum effects significant.
- Understand how the behavior of a Bose gas is different from a classical ideal gas
- Understand when is Bose gas said to be degenerate
- Calculate thermodynamic properties viz equation of state, number density, internal energy, specific heat, Helmholtz free energy and entropy of a Bose gas at $T \neq 0$ involving bose integrals.
- Learn that the non-degenerate Bose gas, for which $\frac{N\lambda^3}{V\mathcal{G}} \ll 1$, behaves like a classical ideal gas.
- Understand through virial expansion of pressure that an ideal bose despite the absence of any interaction gas shows an inherent attractive behavior because of symmetric nature of the multi particle wave function involved
- That as $T \to \infty$, pressure and specific heat approach classical values
- That C_V versus T curve has a maximum at $T = T_C$, where derivative of C_V is discontinuous.
- Learn how with decreasing temperature chemical potential starts increasing and vanishes at a temperature T_C, called Bose-Einstein temperature, which gives rise to Bose-Einstein Condensation. This phase transition is like superfluid transition in liquid helium and condensation of cooper pairs in the BCS theory of superconductivity.
- Underline the fact that Bose- Einstein condensation takes place in a non-interacting gas which happens in momentum space and not in co-ordinate space
- Learn in detail the existence of Bose Einstein condensate in a diluted gas of magnetically confined group of alkaline atoms cooled by evaporation below temperatures of the order of 200 nano kelvin

2. Introduction

In the module on Ideal Bose gas, we focus our attention on the study of an ideal quantum gas made up of indistinguishable particles with integer spin, the so called Bosons, following Bose-Einstein distribution.

$$f(\epsilon) = \frac{1}{\left(e^{\frac{\epsilon-\mu}{k_BT}} - 1\right)} \tag{1}$$

We encounter in nature two kinds of Bose gas, ones with chemical potential $\mu = 0$ and the other with $\mu \neq 0$, but always negative. Examples of the first kind of the Bose gas are Black Body radiation made up of photons, quantized lattice vibrations in a solid called

phonons and quantized spin lattice vibrations called magnons. Taking $\mu = 0$ also implies that constraint applied for the derivation of Bose-Einstein distribution for conservation of particles constituting the system is no longer applicable and these particles can be created by the energy of the heat bath and destroyed by absorption in the heat bath keeping entropy of the heat bath constant. Particles of gas with $\mu = 0$ are made up particles with mass $m = 0$ and are referred to as quasi particles. Such Bose particles have momentum $\vec{p} = \hbar\vec{k}$ and $E = \hbar\omega(\vec{k})$. In the second kind of Bose particles the number of particles constituting the system are conserved and have mass, examples of such system are He^4 and vapours of alkali metals whose atoms constituting the vapours have been cooled down to temperature of the order of micro-kelvin. At $\epsilon = \mu$ Bose-Einstein distribution diverges unless μ is negative. In an ideal Bose gas it is assumed that inter-particle interaction is negligible and quantum effects determine the thermodynamic properties of the system. Second kind of Bose gas also shows a phase transition called Bose-Einstein Condensation.

3. Degenerate Bose Gas

The name degenerate has a special significance here, a system of bosons say photons is said to be degenerate if the average de Broglie wavelength is comparable to or much larger than the average particle distance such that their wave functions overlap and that this overlap cannot be ignored and the system can no longer be treated as classical and must be treated quantum mechanically. It is expected that systems at low T that is at high value of β will be degenerate. The parameter $n\lambda^3$, where n is mean number density and $\lambda = \dfrac{h}{(2\pi m k_B T)^{\frac{1}{2}}}$ thermal de Broglie wave length becomes a very important criteria as to when will quantum effects manifest. For $n\lambda^3 \ll 1$, system behaves like a classical gas. However, as $n\lambda^3 \to 1$ there is a significant departure from classical behavior and system obeys Bose Einstein statistics. Even when gas is ideal the inherent interaction manifests itself in the form of system having symmetric wave function.

4. Thermodynamic Properties of Bose Gas

(a) Equation of State

We had earlier derived the equation of state of an ideal Bose gas

$$\frac{PV}{k_B T} = -\sum_k ln\{1 - e^{(-\beta(\epsilon_k - \mu))} \tag{2}$$

Where $\beta = \dfrac{1}{k_B T}$, μ is chemical potential and $\epsilon_k = \dfrac{\hbar^2 k^2}{2m}$. Replacing $e^{\beta\mu}$ by z and replacing summation over integration equation of state can be obtained as

$$\frac{PV}{k_B T} = -\frac{V}{(2\pi)^3} \int_0^\infty \ln(1 - Ze^{-\beta\epsilon}) \, d^3k \tag{3}$$

However, here if ϵ and μ simultaneously become zero the integrand will diverge at the lower limit, to overcome this difficulty the right procedure shall be to separate out $\epsilon = 0$ term and express (3) as

$$\frac{PV}{k_B T} = -ln(1-z) - \frac{V}{(2\pi)^3} \int ln(1 - Ze^{-\beta\epsilon}) \, d^3k \qquad (4)$$

$$\text{Or} \quad \frac{PV}{k_B T} = -ln(1-z) - \frac{V4\pi}{(2\pi)^3} \int_0^\infty ln(1 - Ze^{-\beta\epsilon}) \, k^2 \, dk \qquad (5)$$

Noting that $\epsilon = \frac{\hbar^2 k^2}{2m}$, $k^2 dk = \frac{1}{2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \epsilon^{\frac{1}{2}} d\epsilon$

$$\text{Or} \quad \frac{PV}{k_B T} = -ln(1-z) - \frac{V4\pi}{2(2\pi)^3} \int_0^\infty ln(1 - Ze^{-\beta\epsilon}) \left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \epsilon^{\frac{1}{2}} d\epsilon \qquad (6)$$

Put $\beta\epsilon = x$, equation (19) can be written as

$$\text{Or} \quad \frac{P}{k_B T} = -\frac{ln(1-z)}{V} - \left(\frac{2\pi m k_B T}{h^2}\right)^{\frac{3}{2}} \left(\frac{4}{\pi}\right)^{\frac{1}{2}} \int_0^\infty ln(1 - Ze^{-x}) \, x^{\frac{1}{2}} dx \qquad (7)$$

$$\text{Or} \quad \frac{P}{k_B T} = -\frac{ln(1-z)}{V} - \frac{1}{\lambda^3}\left(\frac{4}{\pi}\right)^{\frac{1}{2}} \int_0^\infty ln(1 - Ze^{-x}) \, x^{\frac{1}{2}} dx \qquad (8)$$

Where $\frac{1}{\lambda^3} = \left(\frac{2\pi m k_B T}{h^2}\right)^{\frac{3}{2}}$, and $\lambda = \frac{h}{(2\pi m k_B T)^{\frac{1}{2}}}$ is thermal de Broglie wavelength.

Let us see how $I = \int_0^\infty ln(1 - Ze^{-x}) \, x^{\frac{1}{2}} dx$ can be evaluated, it can be integrated by parts

$$I = ln(1 - Ze^{-x})x^{3/2}\Big|_0^\infty - \frac{2}{3}\int_0^\infty \frac{Ze^{-x}}{(1 - Ze^{-x})} x^{\frac{3}{2}} dx \qquad (9)$$

Here the first term vanishes, since first term of the product vanishes at upper limit and second term of the limit vanishes at lower limit.

Therefore, equation (22) reduces to

Or

$$I = -\frac{2}{3}\int_0^\infty \frac{1}{(Z^{-1}e^x - 1)}x^{\frac{3}{2}}dx \tag{10}$$

$$\frac{P}{k_B T} = -\frac{ln(1-z)}{V} + \frac{1}{\lambda^3}\left(\frac{1}{\pi}\right)^{\frac{1}{2}}\frac{4}{3}\int_0^\infty \frac{x^{\frac{3}{2}}}{(Z^{-1}e^x - 1)}dx$$

or

$$= -\frac{ln(1-z)}{V} + \frac{1}{\lambda^3}\,g_{\frac{5}{2}}(z) \tag{11}$$

Where $g_{\frac{5}{2}}(z) = \frac{1}{\Gamma\left(\frac{5}{2}\right)}\int_0^\infty \frac{x^{\frac{3}{2}}}{(Z^{-1}e^x - 1)}dx$, which is a generic integral of the type

$$g_\nu(z) = \frac{1}{\Gamma(\nu)}\int_0^\infty \frac{x^{\nu-1}}{(Z^{-1}e^x - 1)}dx \tag{12}$$

These integrals are known as Bose integrals (See Appendix-A2). And for small z, this integral can be expanded as a convergent series such that

$$g_\nu(z) = z + \frac{z^2}{2^\nu} + \frac{z^3}{3^\nu} - \cdots \tag{13}$$

For $Z \ll 1$, $g_\nu(z) = Z$. As Z increases $g_\nu(z)$ also increases monotonically and over the physical range of interest ($0 < Z \leq 1$) has the maximum value as $Z \to 1$. For $\nu > 1$ then it becomes equal to Riemann zeta function $\zeta(\nu)$.

(b) Number density

Recall

$$N = \sum_\epsilon < n_\epsilon > = \sum_\epsilon \frac{1}{Z^{-1}e^{\beta\epsilon} - 1} \tag{14}$$

Summation can be converted into an integral and replacing $\beta\epsilon = x$ as

$$N = \frac{V}{\lambda^3}\left(\frac{4}{\pi}\right)^{\frac{1}{2}}\int_0^\infty \frac{1}{Z^{-1}e^x - 1}\,x^{\frac{1}{2}}dx \tag{15}$$

However, here if ϵ and μ simultaneously become zero the integrand will diverge at the lower limit, to overcome this difficulty the right procedure shall be to separate out $\epsilon = 0$ term and express (15) as

$$\frac{N}{V} = \frac{1}{V}\left(\frac{Z}{1-Z}\right) + \frac{1}{\lambda^3}\left(\frac{4}{\pi}\right)^{\frac{1}{2}} \int\limits_{0}^{\infty} \frac{1}{Z^{-1}e^x - 1} \; x^{\frac{1}{2}}dx \tag{16}$$

Here first term represents number of particles in the $\epsilon = 0$ state or $p = 0$ state and can be written as N_0. Therefore, (16) can be written as

Or
$$\frac{N - N_0}{V} = \frac{1}{\lambda^3}\left(\frac{4}{\pi}\right)^{\frac{1}{2}} \int\limits_{0}^{\infty} \frac{1}{Z^{-1}e^x - 1} \; x^{\frac{1}{2}}dx = \frac{1}{\lambda^3}g_{\frac{3}{2}}(Z) \tag{17}$$

Let us have a relook at equations (11) and (16)

For $Z \ll 1$, a situation close to classical limit the term $\frac{ln(1-z)}{V}$ in equation (11) and $\frac{z}{(1-Z)V}$ in equation (16) are of the order $\frac{1}{V}$ and hence negligible. But when Z increases and approaches 1, $\frac{Z}{(1-Z)V} = \frac{N_0}{V}$ the number density of particles in zero momentum state is no longer negligible and can be an appreciable fraction of total number of particles $\frac{N}{V}$. Such a situation in which an appreciable fraction of particles accumulate in zero momentum state leads to what is known as Bose Einstein Condensation. However, it must be noted that this is not a condensation in the sense of condensation of a gas of a liquid in real space. It is a condensation in the momentum space.

Nevertheless, in equation (11) $\frac{ln(1-z)}{V}$ remains negligible as Z approaches 1 and this term can be altogether neglected.

(c) Internal Energy

The internal energy of the system is given by

Or
$$U = -\frac{\partial}{\partial \beta}\left(\frac{PV}{k_BT}\right)_{Z,V} = k_BT^2\left[\frac{\partial}{\partial T}\left(\frac{PV}{k_BT}\right)\right]_{Z,V} \tag{18}$$

Or
$$U = k_BT^2\left[\frac{\partial}{\partial T}\left(\frac{V}{\lambda^3}\; g_{\frac{5}{2}}(z)\right)\right]_{Z,V} = Vk_BT^2g_{\frac{5}{2}}(z)\left[\frac{\partial}{\partial T}\left(\frac{1}{\lambda^3}\right)\right] \tag{19}$$
$$= Vk_BT^2g_{\frac{5}{2}}(z)\left[\frac{\partial}{\partial T}\left(\left(\frac{2\pi mk_BT}{h^2}\right)^{\frac{3}{2}}\right)\right]$$
$$= \frac{3}{2}Vk_B\left(\frac{2\pi mk_BT}{h^2}\right)^{\frac{3}{2}} Tg_{\frac{5}{2}}(z) =$$
$$= \frac{3}{2}Vk_B\frac{1}{\lambda^3}\; Tg_{\frac{5}{2}}(z)$$

Once again it is found that ideal Bose Gas satisfies the same relationship between pressure and energy density as is satisfied in the case of ideal fermi gas and classical ideal gas i.e.

Or
$$U = \frac{3}{2} V k_B \frac{1}{\lambda^3} T g_{\frac{5}{2}}(z) = \frac{3}{2} VP \qquad (20)$$

Or
$$P = \frac{2}{3} \frac{U}{V} \qquad (21)$$

(e) Equation of State in Virial Expansion

For small values of Z in equation (11) and (16), first term on the right hand side of each of the equations can be neglected and by noting that

$$\frac{N}{V} = \frac{1}{\lambda^3} g_{\frac{3}{2}}(Z) = \frac{1}{\lambda^3} \left(z + \frac{z^2}{2^{\frac{3}{2}}} + \frac{z^3}{3^{\frac{3}{2}}} - \cdots \right) = \frac{1}{\lambda^3} \sum_{l=1}^{\infty} \frac{z^l}{l^{\frac{3}{2}}} \qquad (22)$$

And

$$\frac{P}{k_B T} = \frac{1}{\lambda^3} g_{\frac{5}{2}}(z) = \frac{1}{\lambda^3} \left(z + \frac{z^2}{2^{\frac{5}{2}}} + \frac{z^3}{3^{\frac{5}{2}}} - \cdots \right) = \frac{1}{\lambda^3} \sum_{l=1}^{\infty} \frac{z^l}{l^{\frac{5}{2}}} \qquad (23)$$

By the process of reverting the series, equation of state in virial expansion can be written. The steps involved are (i) writing z as power series in $\frac{N}{V}$, (ii) substituting it in equation (22) and by equating like powers of $\frac{N}{V}$ on both sides one can obtain coefficients of the power series written in step (i). The value of z thus obtained in (23) one can obtain equation of state in the viral expansion form.

Step (i)
$$z = a_0 + a_1 n + a_2 n^2 + \cdots, \text{where } n = \frac{N}{V}$$

Step (ii)
$$n = \frac{1}{\lambda^3} \left((a_0 + a_1 n + a_2 n^2 + \cdots) + \frac{(a_0 + a_1 n + a_2 n^2 + \cdots)^2}{2^{\frac{3}{2}}} \right.$$
$$\left. + \frac{(a_0 + a_1 n + a_2 n^2 + \cdots)^3}{3^{\frac{3}{2}}} - \cdots \right)$$

On comparing the coefficients on both the sides we get
$$a_0 = 0$$

$$\left(a_1 + \frac{2a_0 a_1}{2^{\frac{3}{2}}}\right)\frac{1}{\lambda^3} = 1, \qquad \therefore a_1 = \lambda^3$$

$$\left(a_2 + \frac{a_1^2}{2^{\frac{3}{2}}}\right)\frac{1}{\lambda^3} = 0$$

$$\therefore a_2 = -\frac{a_1^2}{2^{\frac{3}{2}}} = -\frac{\lambda^6}{2^{\frac{3}{2}}}$$

$$a_3 + \frac{2a_1 a_2}{2^{\frac{3}{2}}} + \frac{a_1^3}{3^{\frac{3}{2}}} = 0$$

$$\therefore a_3 = \frac{\lambda^9}{2^2} - \frac{\lambda^9}{3^{\frac{3}{2}}} = \lambda^9\left[\frac{1}{4} - \frac{1}{3^{\frac{3}{2}}}\right]$$

...

Thus

$$Z = \lambda^3 n - \frac{\lambda^6}{2^{\frac{3}{2}}} n^2 + \lambda^9\left[\frac{1}{4} - \frac{1}{3^{\frac{3}{2}}}\right] n^3 \dots, \tag{24}$$

Substituting Z in equation (23)

Gives

$$\frac{P}{k_B T} = \frac{1}{\lambda^3}\left(z + \frac{z^2}{2^{\frac{5}{2}}} + \frac{z^3}{3^{\frac{5}{2}}} - \cdots\right) \tag{25}$$

$$= \frac{1}{\lambda^3}\left(n\lambda^3 - \frac{\lambda^6}{2^{\frac{5}{2}}} n^2 + \lambda^9\left[\frac{1}{4} - \frac{1}{3^{\frac{3}{2}}} - \frac{1}{8} + \frac{1}{3^{\frac{5}{2}}}\right] n^3 + \cdots\right)$$

$$\frac{P}{k_B T} = n\left(1 - \frac{\lambda^3}{2^{\frac{5}{2}}} n + \lambda^6\left[\frac{1}{8} - \frac{2}{3^{\frac{5}{2}}}\right] n^2 + \cdots\right) \tag{26}$$

$$\frac{PV}{Nk_B T} = \left(1 - \frac{\lambda^3}{2^{\frac{5}{2}}} n + \lambda^6\left[\frac{1}{8} - \frac{2}{3^{\frac{5}{2}}}\right] n^2 + \cdots\right) \tag{27}$$

$$= (a_0 + a_1 n\lambda^3 + a_2 n^2\lambda^6 + \cdots) = \sum_{l=0}^{\infty} a_l(n\lambda^3)^l$$

Equation (26) is the virial expansion of the pressure which is generally written in the form

$$\frac{P}{k_B T} = n + B_2(T)\, n^2 + B_3(T)\, n^3 + \cdots)$$

(28)

$B_j(T)$ are called virial coefficients and these reflect departure from ideal behavior because of some kind of interaction, which is not intermolecular in the present case but arises because of the symmetric nature of the multi-particle wave function of the Bose particles. Note that the second virial coefficient in this case is negative implying that there is an attractive interaction between the Bose particles. This is not so in the case of fermions. As $T \to \infty$, $\lambda \to 0$ and hence pressure of the Bose gas approaches classical result $PV = Nk_B T$.

(d) Specific heat

Let us now calculate specific heat using formula for internal energy using (21)

$$C_V = \frac{\partial U}{\partial T} = \frac{\partial \left(\frac{3}{2} PV\right)}{\partial T} = \frac{3}{2} Nk_B \frac{\partial \left(\frac{PV}{Nk_B}\right)}{\partial T} = \frac{3}{2} Nk_B \frac{\partial \left(\frac{PV}{Nk_B}\right)}{\partial T}$$

(29)

or

$$\frac{C_V}{Nk_B} = \frac{3}{2} \frac{\partial \left(\frac{PV}{Nk_B}\right)}{\partial T} = \frac{3}{2} \frac{\partial \left(\sum_{l=0}^{\infty} a_l (n\lambda^3)^l T\right)}{\partial T} = \frac{3}{2} \sum_{l=0}^{\infty} \left(1 - \frac{3l}{2}\right) a_l (n\lambda^3)^l$$

(30)

or

$$\frac{C_V}{Nk_B} = \frac{3}{2}\left(a_0 + \left(-\frac{1}{2}\right) a_1\, n\lambda^3 + (-2) a_2 (n\lambda^3)^2 + \cdots\right)$$

(31)

or

$$\frac{C_V}{Nk_B} = \frac{3}{2}\left(1 + \left(-\frac{1}{2}\right)\left(-\frac{1}{2^{\frac{5}{2}}}\right) n\lambda^3 + (-2)\left[\frac{1}{8} - \frac{2}{3^{\frac{5}{2}}}\right](n\lambda^3)^2 + \cdots\right)$$

(32)

or

$$\frac{C_V}{Nk_B} = \frac{3}{2}\left(1 + 0.0884\, n\lambda^3 + 0.0066 (n\lambda^3)^2 + \cdots\right)$$

(33)

One can note that as $T \to \infty$, $\lambda \to 0$ and hence specific heat of the Bose gas approaches the classical value $C_V = \frac{3}{2} Nk_B$. At finite temperature specific heat of the gas is greater than the classical value, this implies that C_V versus T graph has negative slope. However since C_V must go to 0 as $T \to 0$, thus (C_V, T) graph must have a maxima. This happens at a critical temperature T_C, where derivative of specific heat is discontinuous.

5. Bose Einstein Condensation

Equations (23) and (33) work well when $n\lambda^3$ is such that temperature is high and λ, the thermal de Broglie wavelength which varies as $T^{-\frac{3}{2}}$ remains small. However as temperature falls we have to go back to equations (11), (17) and (20).

Equation (17) can then be rewritten as

$$\frac{N_e}{V} = \frac{N - N_0}{V} = \frac{1}{\lambda^3} g_{\frac{3}{2}}(Z) = \frac{(2\,\pi m\,k_B T)^{\frac{3}{2}}}{h^3}\,g_{\frac{3}{2}}(Z) \tag{34}$$

Where N_e is the number of particles in the excited state ($\epsilon \neq 0$) and N_0 is the number of particles in the ground state ($\epsilon = 0$). For the range of physical interest i.e. $0 < Z \leq 1$, upper bound occurs when $Z = 1$, at which $g_{\frac{3}{2}}(1) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \equiv \zeta\left(\frac{3}{2}\right) \cong 2.612$.

Therefore, N_e, the total number of particles in all the excited states together has also an bound such that for a given V and T

$$N_e \leq V \frac{(2\,\pi m\,k_B T)^{\frac{3}{2}}}{h^3}\,\zeta\left(\frac{3}{2}\right) \tag{35}$$

There are two possibilities now:

(i)   If actual number of particles in the system is less than this upper bound then all the particles get distributed in the excited states and exact value of Z i.e. fugacity or chemical potential gets determined by taking $N_e \cong N$.

(ii)   If actual number of particles is greater than this limiting value, so the number of particles in excess of this value cannot be distributed in the excited states and must get accommodated in the ground state which has no upper bound and can accommodate an unlimited number of particles, such that

$$N_0 = N - V \frac{(2\,\pi m\,k_B T)^{\frac{3}{2}}}{h^3}\,\zeta\left(\frac{3}{2}\right) \tag{36}$$

And the value of Z can then be obtained using the formula $\frac{Z}{(1-Z)V} = \frac{N_0}{V}$ i.e.

$$Z = \frac{N_0}{N_0 + 1} \cong 1 - \frac{1}{N_0} \tag{37}$$

Which is 1, implying for a given T, $\mu = 0$.

This interesting result opens up a possibility of accommodating a large number of particles in the ground state and is called Bose-Einstein Condensation. And for a given

number of particles $N > N_e$, the system can be seen as a mixture of two phases. One phase made up of particles in the excited states (N_e), $\epsilon \neq 0$ called normal phase and the other made up of N_0 number particles in the ground state called condensed phase.

5.1 Conditions for Onset of Bose-Einstein Condensation

It is now obvious that for Bose-Einstein Condensation to take place the condition can be expressed as

$$N > N_e = V T^{\frac{3}{2}} \frac{(2\pi m \, k_B)^{\frac{3}{2}}}{h^3} \, \zeta\left(\frac{3}{2}\right) \tag{38}$$

It can be expressed alternatively, if we keep N and V fixed as

$$T < T_c = \frac{h^2}{2\pi m k_B} \left(\frac{N}{V \zeta\left(\frac{3}{2}\right)}\right)^{\frac{2}{3}} = \frac{2\pi}{\left(\zeta\left(\frac{3}{2}\right)\right)^{\frac{2}{3}}} \frac{\hbar^2 n^{\frac{2}{3}}}{m k_B} = 3.31 \frac{\hbar^2 n^{\frac{2}{3}}}{m k_B} \tag{39}$$

Where T_c is the transition temperature, which depends on particle density of the system and mass of the particles constituting the system. Fraction of particles in the normal phase $\frac{N_e}{N}$ and fraction of particles in the condensed phase can be written respectively as

$$\frac{N_e}{N} = \left(\frac{T}{T_C}\right)^{\frac{3}{2}} \tag{40}$$

And

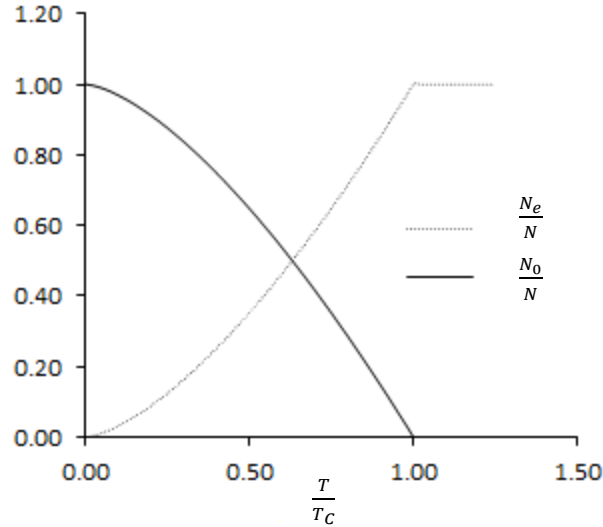$$\frac{N_0}{N} = \frac{N - N_e}{N} = 1 - \left(\frac{T}{T_C}\right)^{\frac{3}{2}} \tag{41}$$

Figure 1 Temprature dependence of the boson fractions in condensed and in normal phase of an ideal Bose gas

Figure 1 shows the plot of fraction of normal phase and fraction of condensed phase plotted against $\frac{T}{T_c}$. It is observed that for $T > T_c$ there is no condensed phase as $\frac{N_0}{N} \to 0$.

5.2 Visualizing Bose-Einstein Condensation in Momentum Space

Equation (16) for number density has been written in energy representation, it can also be written in momentum space as follows

$$N = \left(\frac{Z}{1-Z}\right) + \frac{4\pi V}{h^3} \int_0^\infty \frac{1}{e^{\frac{\beta p^2}{2m}} - 1} \, p^2 dp \tag{42}$$

Below the transition temperature, T_c, as $T \to 0$, the integral in the second term vanishes and the first term gives number of Bosons in the state $p = 0$. So the distribution of the particles can be written as

$$\frac{dN}{dp} = \left(\frac{Z}{1-Z}\right) 2\delta(p) + \frac{4\pi V}{h^3} \frac{p^2}{e^{\frac{\beta p^2}{2m}} - 1} \tag{43}$$

Noting that $2 \int_0^\infty \delta(p) dp = 1$.

On plotting the particle distribution of bose particles in momentum space fior an ideal bose gas below the transition temperature one can visualize the fraction of particles in zero momentum space as given in figure 2.

Figure 2 Momentum distribution of Bose particles in an ideal Bose gas below $T < T_C$, Grey area under the delta function shows the condensate contribution.

5.3 Internal Energy

Recall from equation (20) that

$$U = \frac{3}{2} V k_B \frac{1}{\lambda^3} T g_{\frac{5}{2}}(z) \tag{44}$$

And from equation (17)

$$\frac{N - N_0}{V} = \frac{1}{\lambda^3} g_{\frac{3}{2}}(Z) \tag{45}$$

or

$$N - N_0 = N_e = \frac{V}{\lambda^3} g_{\frac{3}{2}}(Z) \tag{46}$$

or
$$\frac{V}{\lambda^3} = \frac{N_e}{g_{\frac{3}{2}}(Z)} \qquad (47)$$

Therefore, equation (44) can be written as

or
$$U = \frac{3}{2} \, k_B T N_e \, \frac{g_{\frac{5}{2}}(z)}{g_{\frac{3}{2}}(Z)} \qquad (48)$$

Here the factor $\dfrac{g_{\frac{5}{2}}(z)}{g_{\frac{3}{2}}(Z)}$ gives the deviation of internal energy from the corresponding

classical value. Noting from equation (40) that $\dfrac{N_e}{N} = \left(\dfrac{T}{T_C}\right)^{\frac{3}{2}}$, (48) can be written as

or
$$U = \frac{3}{2} \, N k_B T \frac{N_e}{N} \frac{g_{\frac{5}{2}}(z)}{g_{\frac{3}{2}}(Z)} = \frac{3}{2} \, N k_B T \left(\frac{T}{T_C}\right)^{\frac{3}{2}} \frac{g_{\frac{5}{2}}(z)}{g_{\frac{3}{2}}(Z)} = \left(\frac{3}{2} \, N k_B T_c\right) \left(\frac{T}{T_C}\right)^{\frac{5}{2}} \frac{g_{\frac{5}{2}}(z)}{g_{\frac{3}{2}}(Z)} \qquad (49)$$

For $< T_C$, the fugacity Z=1. Equation (49) can be written as

or
$$U = \left(\frac{3}{2} \, N k_B T_c\right) \left(\frac{T}{T_C}\right)^{\frac{5}{2}} \frac{g_{\frac{5}{2}}(1)}{g_{\frac{3}{2}}(1)} = \left(\frac{3}{2} \, N k_B T_c\right) \left(\frac{T}{T_C}\right)^{\frac{5}{2}} \frac{\zeta_{\frac{5}{2}}}{\zeta_{\frac{3}{2}}} \qquad (50)$$

Using the value of Riemann-Zeta function from appendix, we can write

or
$$U = \left(\frac{3}{2} \, N k_B T_c\right) \left(\frac{T}{T_C}\right)^{\frac{5}{2}} \frac{1.341}{2.612} = \left(\frac{3}{2} \, N k_B T_c\right) \left(\frac{T}{T_C}\right)^{\frac{5}{2}} 0.51 = 0.76 \left(N k_B T_c\right) \left(\frac{T}{T_C}\right)^{\frac{5}{2}} \qquad (51)$$

In this region internal energy $U \propto T^{\frac{5}{2}}$. At $T = T_c$ the energy is approximately half that of an ordinary gas at the same temperature.

For $> T_c$, $Z < 1$ and all particles are in excited states i.e. $N = N_e$

or
$$U = \frac{3}{2} \, N k_B T \, \frac{g_{\frac{5}{2}}(z)}{g_{\frac{3}{2}}(Z)} \qquad (52)$$

For $Z < 1$, $\dfrac{g_{\frac{5}{2}}(z)}{g_{\frac{3}{2}}(Z)} < 1$ and, therefore, in this region internal energy is always less than the classical value of $\dfrac{3}{2}\ Nk_BT$.

5.4 Specific Heat at Constant Volume of Bose Einstein Condensate

Specific heat at constant volume is given by

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V \tag{53}$$

For $< T_c$, it gives using (51)

or

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{5}{2}\,(\,0.76\,)(\,Nk_B\,)\left(\frac{T}{T_C}\right)^{\frac{3}{2}} = 1.9\,Nk_B\left(\frac{T}{T_C}\right)^{\frac{3}{2}} \tag{55}$$

Therefore, $C_V \propto T^{\frac{3}{2}}$.

At $= T_c$, we have

or

$$C_V = 1.9\,Nk_B \tag{56}$$

This value is much higher than the classical value of $\dfrac{3}{2}\,Nk_B$.

At $T > T_C$, differentiating (52) we get

or

$$C_V == \ \frac{3}{2}\ Nk_B\frac{\partial}{\partial T}\left(\ T\ \ \frac{g_{\frac{5}{2}}(z)}{g_{\frac{3}{2}}(Z)}\right) \tag{57}$$

This differentiation requires knowledge of temperature derivative of Z

From (47) it is clear that $g_{\frac{3}{2}}(Z) \propto T^{-\frac{3}{2}}$

Therefore,

$$\frac{\partial g_{\frac{3}{2}}(Z)}{\partial T} = -\frac{3}{2T}\ g_{\frac{3}{2}}(Z) \tag{58}$$

And recalling the recurrence relation $z\dfrac{\partial g_v(z)}{\partial z} = g_{v-1}(z),$

$$z \frac{\partial g_{\frac{3}{2}}(z)}{\partial z} = g_{\frac{1}{2}}(z) \tag{59}$$

Using (58) and (59) we get

$$\frac{1}{Z}\left(\frac{\partial Z}{\partial T}\right)_{V,N} = -\frac{3}{2T} \frac{g_{\frac{3}{2}}(Z)}{g_{\frac{1}{2}}(z)} \tag{60}$$

Carrying out temperature derivative in equation (57), we have

$$C_V == \frac{3}{2} Nk_B \left(\frac{g_{\frac{5}{2}}(z)}{g_{\frac{3}{2}}(Z)} + T \left(\frac{g_{\frac{3}{2}}(Z) g'_{\frac{5}{2}}(z) - g_{\frac{5}{2}}(z) g'_{\frac{3}{2}}(Z)}{g_{\frac{3}{2}}^2(Z)} \right) \frac{\partial}{\partial T}(Z) \right) \tag{61}$$

Or

$$C_V == \frac{3}{2} Nk_B \left(\frac{g_{\frac{5}{2}}(z)}{g_{\frac{3}{2}}(Z)} + T \left(\frac{g_{\frac{3}{2}}(Z) g_{\frac{3}{2}}(z) - g_{\frac{5}{2}}(z) g_{\frac{1}{2}}(Z)}{g_{\frac{3}{2}}^2(Z)} \right) Z \frac{\partial}{\partial T}(Z) \right) \tag{62}$$

Or

$$C_V == \frac{3}{2} Nk_B \left(\frac{g_{\frac{5}{2}}(z)}{g_{\frac{3}{2}}(Z)} + T \left(\frac{g_{\frac{3}{2}}(Z) g_{\frac{3}{2}}(z) - g_{\frac{5}{2}}(z) g_{\frac{1}{2}}(Z)}{g_{\frac{3}{2}}^2(Z)} \right) \right) \left(-\frac{3}{2T} \frac{g_{\frac{3}{2}}(Z)}{g_{\frac{1}{2}}(z)} \right) \tag{63}$$

Or

$$C_V == \frac{3}{2} Nk_B \left(\frac{g_{\frac{5}{2}}(z)}{g_{\frac{3}{2}}(Z)} - \frac{3}{2} \left(\frac{g_{\frac{3}{2}}(Z)}{g_{\frac{1}{2}}(z)} - \frac{g_{\frac{5}{2}}(Z)}{g_{\frac{3}{2}}(Z)} \right) \right) \tag{64}$$

Or

$$C_V = Nk_B \left(\frac{15}{4} \frac{g_{\frac{5}{2}}(z)}{g_{\frac{3}{2}}(Z)} - \frac{9}{4} \left(\frac{g_{\frac{3}{2}}(Z)}{g_{\frac{1}{2}}(Z)} \right) \right) \tag{65}$$

Here we note that term in the parentheses on the right hand side decreases till in the limit
$\rightarrow \mathbf{0}$,

Or

$$C_V = Nk_B \left(\frac{15}{4} - \frac{9}{4} \right) = \frac{3}{2} Nk_B \tag{66}$$

For $T = T_C$, i.e. in the limit $Z \to 1$, $g_{\frac{1}{2}}(z)$ diverges and the second term vanishes and the contribution comes only from first term which is

Or

$$C_V = Nk_B \left(\frac{15}{4} \frac{g_{\frac{5}{2}}(1)}{g_{\frac{3}{2}}(1)} \right) = Nk_B \left(\frac{15}{4} \frac{\zeta\left(\frac{5}{2}\right)}{\zeta\left(\frac{3}{2}\right)} \right) = 1.925 \, Nk_B \tag{67}$$
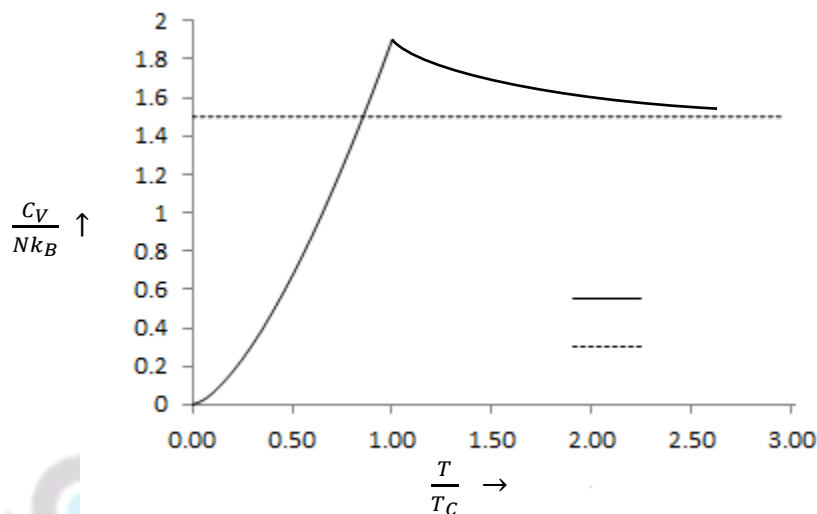
It is the same result as given by (56).

Figure 3 Specific heat vs $\frac{T}{T_c}$ of an ideal Bose gas

Figure 3 gives a plot of specific heat of an ideal Bose gas against reduced temperature $\frac{T}{T_c}$. It is a continuous curve at $T = T_c$, however, its slope is discontinuous. Furthermore, since below T_c and above T_c the properties of the ideal Bose gas are different, it may be concluded that there is a phase transition. There is a real system corresponding to He^4, a bosonic system, to which the results of specific heat resemble, though that system is far from an ideal Bose gas behavior. The transition point there is called lambda transition, since the specific heat curve of this He^4 bosonic system resembles the greek letter λ.

5.5 Pressure

From equation (20), we know

$$PV = \frac{2}{3}U = V \frac{k_B T}{\lambda^3} g_{\frac{5}{2}}(z) \tag{68}$$

For $< T_c$, $z = 1$, $g_{\frac{5}{2}}(z) = 1.341$ and, therefore,

$$P = 1.341 \frac{k_B T}{\lambda^3} = 1.341 \frac{k_B T}{\lambda^3} = 1.341 k_B T \frac{(2 \pi m \, k_B T)^{\frac{3}{2}}}{h^3} \quad (69)$$

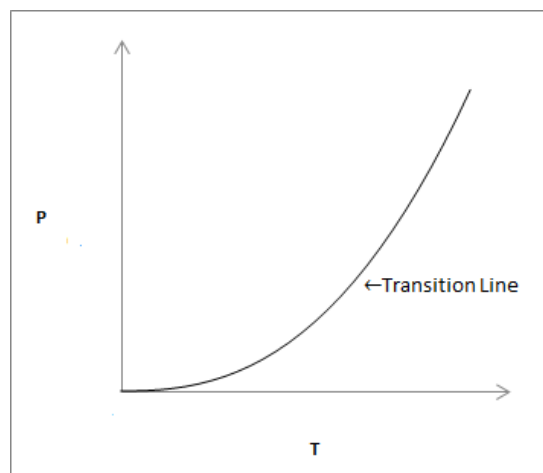In this region we see that pressure is independent of V, implying and P is $\propto T^{\frac{5}{2}}$.

Figure 4 PT diagram of the ideal Bose gas. $PT^{\frac{5}{2}} = constant$. Condensed phase lies on the line and below it is the normal gas phase

At $T = T_c$,

$$P = 1.341 \frac{(2 \pi m)^{\frac{3}{2}}}{h^3} (k_B T_C)^{\frac{5}{2}} \quad (70)$$

Using (39), $k_B T_c = \frac{h^2}{2\pi m} \left(\frac{N}{V \zeta(\frac{3}{2})} \right)^{\frac{2}{3}}$ in (69), we get

$$P = \frac{1.341}{\zeta\left(\frac{3}{2}\right)} \frac{N}{V} k_B T_C = \frac{1.341}{2.612} \frac{N}{V} k_B T_C = 0.5133 \frac{N}{V} k_B T_C \quad (71)$$

$$\frac{PV}{Nk_B} \frac{1}{T_C} = 0.5133 \quad (72)$$

It shows that temperature exerted by an ideal Bose gas at $T = T_c$ is half the pressure exerted by a classical ideal gas.

For $> T_c$,

$$P = \frac{k_B T}{\lambda^3} g_{\frac{5}{2}}(z) = \frac{k_B T}{\frac{V}{N}} \frac{g_{\frac{5}{2}}(z)}{g_{\frac{3}{2}}(z)} \qquad (73)$$

or

$$P = \frac{N k_B T}{V} \frac{g_{\frac{5}{2}}(z)}{g_{\frac{3}{2}}(z)} \qquad (74)$$

Since The ratio of the two functions is always less than unity, pressure is once again less than the classical ideal gas. As $T \to \infty$, $Z \to 0$ and pressure asymptotically approaches classical value.

5.6 Entropy

For $T \leq T_c$, using (55) the expression for C_V, we can calculate entropy by integrating

$$S = \int_0^T \frac{C_V}{T} \, dT = \frac{2}{3} 1.9 k_B N \left(\frac{T}{T_c}\right)^{\frac{3}{2}} = 1.28 \, k_B N_e \qquad (75)$$

Where we have used $N_e = N \left(\frac{T}{T_c}\right)^{\frac{3}{2}}$. In this region normal phase and condensed phase both are present but it is the particles N_e in the normal phase which contribute to the entropy of the system. Contribution to Entropy by the particles N_0 in condensed phase is zero. At $0^o K$ all particles are in ground state and the entropy of the system $S = 0$.

Also recalling the thermo-dynamical relation

$$G = U - TS + PV = \mu N \qquad (76)$$

$$\frac{S}{k_B N} = \frac{U + PV}{k_B T N} - \frac{\mu}{k_B T} \qquad (77)$$

And noting that for $T < T_C$, $\mu = 0$,

Therefore,

$$\frac{S}{k_B N} = \frac{U + PV}{k_B TN} = \frac{5}{3}\frac{U}{k_B TN} = \frac{5}{2}\frac{PV}{k_B TN} = \frac{5}{2}\frac{V}{N\lambda^3}\zeta\left(\frac{5}{2}\right) = \frac{5}{2}\frac{\zeta\left(\frac{5}{2}\right)}{\zeta\left(\frac{3}{2}\right)} \tag{78}$$

Since $N = N_e$

$$S = k_B N_e \frac{5}{2}\frac{\zeta\left(\frac{5}{2}\right)}{\zeta\left(\frac{3}{2}\right)} \tag{79}$$

Or $S \propto N_e$.

For $T > T_C$,

$$\frac{S}{k_B N} = \frac{U + PV}{k_B TN} - \frac{\mu}{k_B T} = \frac{5}{2}\frac{PV}{k_B TN} - \ln(Z) = \frac{5}{2}\frac{g_{\frac{5}{2}}(Z)}{g_{\frac{3}{2}}(Z)} - \ln(Z) \tag{80}$$

5.7 Isotherms for an Ideal Bose Gas

We now consider isotherms of an ideal Bose gas. According to equation (28), at constant temperature pressure P varies as $\left(\frac{1}{V} + O(V^2)\right)$ for $V > V_C$, while for $V < V_C(T)$ it is independent of V. The isotherms are plotted in figure 5 below.
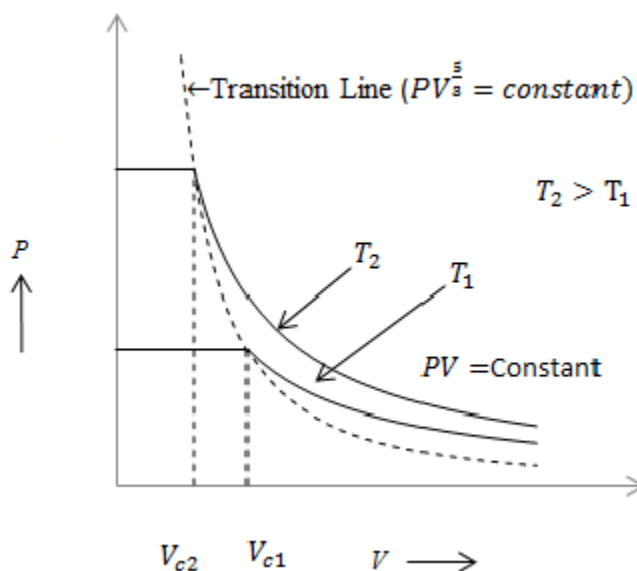

Figure 5 Isotherms for an ideal Bose gas in a $P - V$ diagram. Dashed line $\left(P \propto \frac{1}{V^{\frac{5}{3}}}\right)$ is the transition line, region to the left of this line is condensation region.

V_{C1} and V_{C2} are the volumes where the isotherms for T_1 and T_2 touch the transition line. For $V < V_c$, pressure remains constant, where as for $V > V_c$, as volume decreases pressure increase till the isotherm touches the transition line. The region below transition line is a co-existence region, having both the condensed phase and the normal phase. As the volume decrease the number of particles in the condensed phase increases. It can also be seen from the isotherms that pressure in the co-existence region is determined by the temperature alone. For $T_2 > T_1$, $P_2 > P_1$. Also in the coexistence region, since pressure is solely dependent on temperature. It corresponds to first order phase transition.

5.8 Adiabatic Relations of Ideal Bose Gas

For a reversible adiabatic process, entropy is constant. Therefore, form (75) for $< T_C$ $S = 1.28 \, k_B N_e$, N_e is constant. From (47) $\frac{V}{\lambda^3} = \frac{N_e}{g_{\frac{3}{2}}(Z)}$, we have,

$$\frac{V}{\lambda^3} = \frac{N_e}{\zeta_{\frac{3}{2}}(1)} = constant \tag{81}$$

Hence

$$\frac{V}{T^{-\frac{3}{2}}} = constant \tag{82}$$

On the Other hand for $T > T_C$, from equation (80) it follows that fugacity z and $g_{\frac{5}{2}}(Z)$ are both constant. Hence from equation (73)

$$\frac{P}{T^{\frac{5}{2}}} = constant \tag{83}$$

Eliminating, T from (82) and (83) we get adiabatic equation of ideal Bose gas.

$$PV^{\frac{5}{3}} = constant \tag{84}$$

6 Bose Einstein Condensation: Experimental Realisation

In the study of Ideal Bose gas model, it is natural to look for its manifestation in nature in the form of Bose Einstein Condensation. The real challenge in observing this lies in getting a gas of non-interacting bosons. This requires average particle- particle distance (r_d) of the boson gas to be greater than the range of particle-particle interaction (r_i). Typical range of particle –particle interaction is of the order of s-wave scattering length (a_s). Also the thermal de-Broglie wave length (λ) should be much greater than the particle- particle distance (r_d). So when we are able to satisfy the criteria $\lambda \gg r_d \gg r_i \sim a_s$ we can treat the bosonic system as an ideal Bose gas.

Its first signatures were thought to be not in the so called non-interacting gas but in a quantum liquid in the form of liquid Helium-4 which is a collection of bosons and has

an extraordinary property of superfluidity which was to begin with thought to be attributed to Bose Einstein Condensation.

It was only lately, in 1995, that Bose Einstein Condensation was demonstrated in the dilute gas of Rubidium -87 atoms when its vapours were cooled below $170nK$. We will look at these two experimental realisations of Bose Einstein Condensations in the Following

6.1 Liquid 4_{He}

Helium is a very interesting element, which has two isotopes 4_{He} and 3_{He}. 4_{He} is most abundant and 3_{He} the least abundant. Helium was found to be a liquid form only at extremely low temperature at 4 K downwards. 4_{He} atoms are having even number of spin $\frac{1}{2}$ particles are bosons and 3_{He} atoms having odd number of particles are fermions.

Liquid 4_{He} has an interesting phase transition called λ transition from phase called He I to a phase called Helium II as shown in figure 6. The later phase is a superfluid phase. This transition takes place at T_λ=2.18 K.
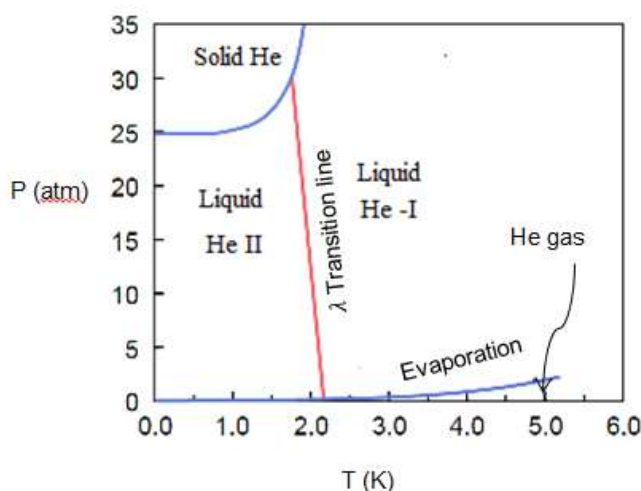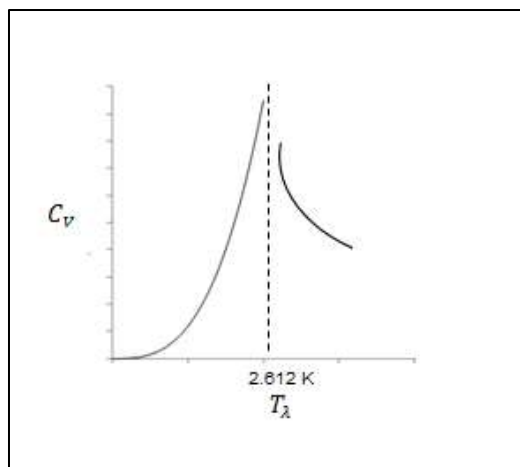

Figure 6 Phase diagram of 4_{He}

Figure 7 Specific Heat for Superfield Helium II

Interestingly, if we take the mass of 4_{He} atom to be $m = 6.65 \times 10^{-4}$ g and use the formula for calculating the transition temperature with specific volume v_λ, and $g_{\frac{3}{2}}(1)=2.612$ one gets $T_c=3.12$. This number is very close to T_λ. But if we look at the experimental $C_V - T$ curve for 4_{He}, figure 7, it has a logarithmic divergence and is a departure from finite C_V at T_c in an ideal Bose gas. And, therefore, this transition need to be looked into by including rather strong interaction among the 4_{He} atoms. This has been a question which was pondered over by Landau and Feynman particularly in the context of superfluidity shown by 4_{He}. We will look at thus system in more details in the next module.

6.2 Ultra-Cold Atomic Gases

Bose Einstein Condensate in Ultra Cold atomic gases of ^{87}Rb isotope by Carl Wiemann's group in colorado was the first demonstrations of a pure Bose Einstein condensate in Matter in 1995. In quick succession Ketterle's group in MIT Massachusetts demonstrated Bose Einstein Condensate of ^{23}Na. This was made possible by convergence of many innovative technologies to magnetically trap a group of thousands of atoms first cooled by counter propagating laser beams and further cooled by evaporative cooling. Magnetic trapping made sure that a desirable number of Bosons are available in a specific volume.

Alkali atoms chosen for this purpose have valence one leading to half integer spin, $\frac{1}{2}\hbar$, But for Bose Einstein condensation one needs integral spin. Since nuclear spin of alkali atoms (^{87}Rb, ^{23}Na, ^{7}Li)is also half integer spin, $\frac{3}{2}\hbar$, so the first requirement which needs to be fulfilled is the coupling of the electronic spin of the electrons with the nuclear spin. This requires total spin $\vec{J} = \vec{I} + \vec{S}$, in the range of $|I + S|$ and $|I - S|$, where $\vec{I}$ and $\vec{S}$ are respectively nuclear and electronic spins of the atoms. Therefore, total spin of the alkali atoms is either $2\hbar$ or $1\hbar$. This also requires that isotopes of the alkali atoms chosen should be such that both atomic number and mass number should be odd. So that $Z + A$ is even and atom as a whole has even number of fermions so that total $\vec{J}$ is integer and hence a Boson.

It is worth estimating T_C for say ^{87}Rb using formula $T_c = 3.31 \frac{\hbar^2 n^{\frac{2}{3}}}{mk_B}$. ^{87}Rb has a typical value of density of the order of n$\sim 10^{18} m^{-3}$. Using $m = (37 m_p + 50 m_n + 37 m_e)$ one gets $T_c \sim 18.28 \, nK \sim 10^{-7} K$. Furthermore, hyperfine interaction energy for Rb is about 0.3 K, meaning thereby that below about 0.1 K, coupling between nuclear spin and electronic spin survives. Thus at a temperature at about $10^{-7} K$ or below, Bose Einstein Condensation will happen in ^{87}Rb can be experimentally observed.

Steps to experimentally produce Bose Einstein Condensate:

(a) First pure bosons in the form of particular isotopes of say ^{87}Rb are taken and put in vacuum.

(b) The atoms so taken are cooled to fractions of a ^{0}K.

(c) Then these cooled atoms are put in a magnetic trap which is pair of co-axial anti Helmholtz coils with current flowing in opposite direction leading to a quadrupole field, such that in the central region of two coils magnetic field is zero. This region acts as a trap forming a plus like configuration with opposite points having same magnetic polarity.

(d) A laser with a precisely calculated wavelength tuned slightly below the resonant frequency of the atoms in the trap is beamed on the atoms. Atoms which are stationary are off the resonance and do not absorb an incident photon. On the other hand moving atoms are Doppler shifted on resonance to the laser beam which is moving in a direction opposite to the velocity vector of the atom. These atoms absorb photons from that direction and then reemits in the random directions, causing a net momentum impulse in a direction opposite to the direction of motion. This results in slowing down of the atoms and hence in lowering the temperature. In fact the experimental set up uses three sets of counter propagating beams oriented along Cartesian axis. This step has limit to cooling because of recoil limit when the atoms acquire a minimum momentum of the order of momentum of the photons of the laser beam. This step cools the atoms to a temperature of the order of 10^{-6}K. The real challenge in this step is to keep the laser tuned to desirable frequency.

(e) Further cooling is caused by evaporative cooling, which essentially allows atoms moving faster to escape from the trap leaving the lowest energy atoms behind. A stage comes when the only atoms left behind are the ground state atoms and they coalesce into a Bose Einstein condensate, which behaves like a single superatom. Rubidium atoms because of highest mass could go into the ground state at the highest temperature.

(f) The next step was to detect the Bose Einstein Condensate. This is achieved by suddenly switching off the trapping forces, allowing the super atom to expand. Colder the atoms slower is the expansion of the super atom.Using a resonant laser light an image of this expansion can be captured. The famous picture shown in figure 8 captures the first Bose Einstein Condensate.
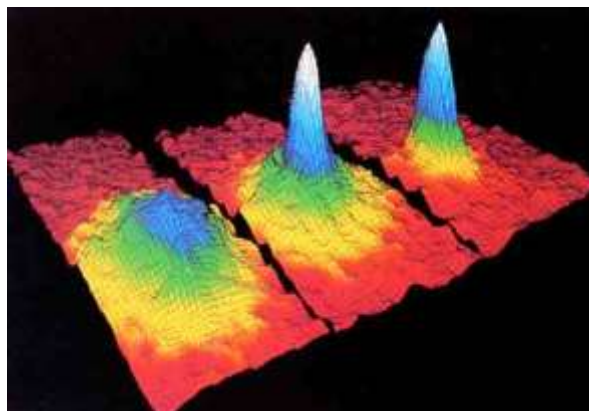
Figure 8 Sequence of Bose Einstein Condensation in Rubidium 87. Picture shows atomic distribution before BEC, at the start start of BEC and after full BEC.
(*Source:https://upload.wikimedia.org/wikipedia/commons/thumb/a/af/Bose_Einstein_condensate.png/238px-Bose_Einstein_condensate.png*)

5. Summary

In this module we have learnt

- about broad learning goals of this e-course in statistical mechanics and what are the pre-requisites for appreciating it.
- that despite the laws governing the constituent particles are well known, yet coming together of a large number of particles makes the understanding of such a system possible only by statistical means, revealing new laws unheard of in the realm of individual particle behavior.
- that new laws are probabilistic in nature and give an average behavior of the properties of the system.
- that how statistical approach relates to thermo-dynamical physical quantities via boltzmann's famous relation for entropy and thus opens a pathway to link statistical mechanics with thermodynamics(a very well established phenomenological theory)?
- that statistical mechanics deals with a many particle system with a challenge of handling complexity of them all together, usually with very meager information available at any point of time.
- that the long list of its applications provide a wide sweep and utility of statistical mechanics not only in understanding physical phenomenon, but also phenomenon encountered in variety of diverse fields in biology, economics, ecology, information theory and computer science.

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Appendix

A1 Bose Integrals

The integrals of the following type encountered in this module are called Bose integrals

$$g_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1}}{(Z^{-1}e^x - 1)} dx$$

It has following interesting properties:

(a) For small Z, the factor $\frac{1}{(Z^{-1}e^x - 1)}$ in the integrand can be written as $Ze^{-x}(1 - Ze^{-x})^{-1}$ and can be expanded as given below

$$Ze^{-x}(1 - Ze^{-x})^{-1} = Ze^{-x}\left(1 + (-1)(-Ze^{-x}) + \frac{(-1)(-2)}{2!}(-Ze^{-x})^2 + \frac{(-1)(-2)(-3)}{3!}(-Ze^{-x})^3 + \cdots\right)$$

$$= (Ze^{-x} + (Ze^{-x})^2 + (Ze^{-x})^3 + (Ze^{-x})^4 + \cdots = \sum_{l=1}^\infty (-1)^{l-1}(Ze^{-x})^l$$

$$g_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty x^{\nu-1} \sum_{l=1}^\infty (Ze^{-x})^l \, dx$$

Let us now look at the lth term of the integral

$$Z^l \int_0^\infty x^{\nu-1} e^{-lx} \, dx$$

Put $lx = y$ in the above integral, we get $dx = \frac{dy}{l}$ and it takes the form

$$\frac{Z^l}{l^\nu} \int_0^\infty (y)^{\nu-1} e^{-y} \, dy = \frac{Z^l}{l^\nu} \Gamma(\nu)$$

Therefore,

$$g_v(z) = \sum_{l=1}^{\infty} \frac{Z^l}{l^v} = z + \frac{z^2}{2^v} + \frac{z^3}{3^v} + \cdots$$

(b) The integral $g_v(z)$ satisfies the following recurrence relation which can be easily verified from the series given above

$$z\frac{\partial g_v(z)}{\partial z} = g_{v-1}(z)$$

(c) For $Z \ll 1$, the function $g_v(z)$ behaves as Z. Also $g_v(z)$ is a monotonically increasing function of z. Since largest value of z which is of physical interest is 1, then for $v > 1$, $g_v(z)$ approaches the Riemann-Zeta function $\zeta(v)$

$$g_v(1) = \sum_{l=1}^{\infty} \frac{1}{l^v} = \zeta(v) \text{ for } v > 1$$

Table of Riemmann-Zeta functions $\zeta(v)$ for various values of $v > 1$

v	$\zeta(v)$	
	Calculated analytically	Calculated numerically, rounded to three decimal places
$\frac{3}{2}$		2.612
2	$\frac{\pi^2}{6}$	1.645
$\frac{5}{2}$		1.341
3		1.202
4	$\frac{\pi^4}{90}$	1.082